SPACES OF ANALYTIC FUNCTIONS OF HARDY-BLOCH TYPE

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Abstract. For $0 < p \le \infty$ and $0 < q \le \infty$, the space of Hardy-Bloch type $\mathscr{B}(p,q)$ consists of those functions f which are analytic in the unit disk $\mathbb{D}$ such that $(1-r)M_p(r,f') \in L^q(dr/(1-r))$. We note that $\mathscr{B}(\infty,\infty)$ coincides with the Bloch space $\mathscr{B}$ and that $\mathscr{B} \subset \mathscr{B}(p,\infty)$, for all p. Also, the space $\mathscr{B}(p,p)$ is the Dirichlet space $\mathcal{D}^p_{p-1}$.

We prove a number of results on decomposition of spaces with logarithmic weights which allow us to obtain sharp results about the mean growth of the $\mathscr{B}(p,q)$-functions. In particular, we prove that if f is an analytic function in $\mathbb{D}$ and $2 \le p < \infty$, then the condition $M_p(r,f') = \mathrm{O}\left((1-r)^{-1}\right)$, as $r \to 1$, implies that $M_p(r,f) = \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)^{1/2}\right)$, as $r \to 1$. This result is an improvement of the well known estimate of Clunie and MacGregor and Makarov about the integral means of Bloch functions, and it also improves the main result in a recent paper by Girela and Peláez. We also consider the question of characterizing the univalent functions in the spaces $\mathscr{B}(p,2)$, $0 < p < \infty$, and in some other related spaces and give some applications of our estimates to study the Carleson measures for the spaces $\mathscr{B}(p,2)$ and $\mathcal{D}^p_{p-1}$.

1. Introduction and statements of the main results

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$. If $0 < r < 1$ and f is an analytic function in $\mathbb{D}$ (abbreviated $f \in H(\mathbb{D})$) we set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right)^{1/p}, \quad I_p(r,f) = M_p^p(r,f),\ (0 < p < \infty),$$

$$M_\infty(r,f) = \sup_{0\le t\le 2\pi} |f(re^{it})|.$$

For $0 < p \le \infty$ the Hardy space H^p consists of those functions $f \in H(\mathbb{D})$ for which

$$||f||_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty.$$

We refer to [9] for the theory of Hardy spaces. For $0 < p < \infty$, the Bergman space A^p is the set all $f \in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f(z)|^p\,dA(z) < \infty$$

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where $dA(z) = dx\,dy = r\,dr\,d\theta$ is the Lebesgue area measure. We mention [11] and [21] as general references for the theory of Bergman spaces.

For $0 < p \le \infty$ and $0 < q \le \infty$, we shall write $\mathscr{B}(p,q)$ for the space of those $f \in H(\mathbb{D})$ such that

$$\begin{aligned} K_{p,q}(f) &\stackrel{\text{def}}{=} \left(\int_0^1 M_p^q(r,f')(1-r)^{q-1}\,dr\right)^{1/q} < \infty, \qquad &\text{if } q<\infty, \\ K_{p,\infty}(f) &\stackrel{\text{def}}{=} \sup_{0<r<1}(1-r)M_p(r,f') < \infty, &\text{if } q=\infty. \end{aligned} \tag{1.1}$$

The spaces $\mathscr{B}(p,p)$ $(0 < p < \infty)$ are the Dirichlet spaces $\mathcal{D}^p_{p-1}$ which have been extensively studied in [4, 16, 35, 37].

There is a close connection between the spaces $\mathscr{B}(p,q)$ and the Hardy spaces. Let us remark that $H^2 = \mathscr{B}(2,2) = \mathcal{D}^2_1$. Hardy and Littlewood proved in [20] that

$$H^p \subset \mathscr{B}(p,2), \quad \text{if } 0 < p \le 2, \tag{1.2}$$

and

$$\mathscr{B}(p,2) \subset H^p, \quad \text{if } 2 \le p < \infty. \tag{1.3}$$

On the other hand, we have

$$\boxed{\mathcal{D}^p_{p-1} \subset H^p, \quad \text{if } 0 < p \le 2,} \quad \text{and} \quad \boxed{H^p \subset \mathcal{D}^p_{p-1}, \quad \text{if } 2 \le p < \infty.} \tag{1.4}$$

For $2 \le p < \infty$, this is a classical result of Littewood and Paley [23]. For the case $0 < p < 2$ see, e.g., [35] and [28].

The Bloch space $\mathscr{B}$ (cf. [1]) consists of those functions $f \in H(\mathbb{D})$ for which

$$M_\infty(r,f') = \mathrm{O}\left(\frac{1}{1-r}\right), \quad \text{as } r \to 1.$$

With the terminology just introduced, we have $\mathscr{B} = \mathscr{B}(\infty,\infty)$ and $\mathscr{B} \subset \mathscr{B}(p,\infty)$, for all $p > 0$. Clunie and MacGregor [8] and Makarov [24] proved the following.

Theorem A. If $f \in \mathscr{B}$, then for all $p < \infty$ we have

$$M_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right), \quad \text{as } r \to 1.$$

For $0 < p < \infty$, the space $\mathscr{B}(p,\infty)$ was called $\mathcal{F}_p$ by Girela and Peláez in [17] where the following extension of Theorem A was proved.

Theorem B. If $f \in \mathscr{B}(p,\infty)$, then

$$M_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{\beta}\right), \quad \text{as } r \to 1, \tag{1.5}$$

where

(i) $\beta = 1/p$, for $0 < p \le 2$,
(ii) β is any number greater than $1/2$, for $2 < p < \infty$.

It was proved in [17] that if $p < 2$, then the exponent $\beta = 1/p$ is best possible, but the question of whether one can take $\beta = 1/2$ in (ii) remained open. Here we answer this question affirmatively by proving the following improvement of Theorem B.

Theorem 1.1. *If $f \in \mathscr{B}(p,\infty)$, $2 < p < \infty$, then*

$$M_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right), \quad \text{as } r \to 1. \tag{1.6}$$

This theorem does not hold for $p = \infty$ as is shown by the example $f(z) = \log(1/(1-z))$. In this case Korenblum [22] proved the following result:

Theorem C. If $f \in \mathscr{B}$, then

$$\|f_r\|_{BMOA} = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right), \quad \text{as } r \to 1. \tag{1.7}$$

Here $BMOA$ denotes the space of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\mathbb{T} = \partial\mathbb{D}$ (cf. [2], [14] and [15]), and

$$f_r(z) = f(rz), \quad z \in \mathbb{D},\ 0 < r < 1.$$

Since $M_p(r,f) \le C_p\|f_r\|_{BMOA}$, $p < \infty$, Theorem C improves Theorem A.

We shall see in Section 2 that Theorem 1.1 and Theorem C can be deduced from two known results, one of which is due to Hardy and Littlewood. However, we will give an approach based on decomposition of spaces with logarithmic weights which will be presented in sections 4, 5 and 6. This enables us to extend Theorem 1.1 to the "integrated" Bloch–Hardy spaces $\mathscr{B}(p,q)$. Namely, in Section 7 we shall prove the following result.

Theorem 1.2. *If $2 < p < \infty$, $2 < q \le \infty$ and $f \in \mathscr{B}(p,q)$, then $I_{p,q}(f) < \infty$, where*

$$\begin{aligned} I_{p,q}(f) &\overset{def}{=} \left(\int_0^1 M_p^q(r,f)\left(\log\frac{2}{1-r}\right)^{-q/2}\frac{dr}{1-r}\right)^{1/q}, \quad \textit{if } q < \infty, \\ I_{p,\infty}(f) &\overset{def}{=} \sup_{0<r<1} M_p(r,f)\left(\log\frac{2}{1-r}\right)^{-1/2}. \end{aligned} \tag{1.8}$$

Furthermore, there exists a positive constant $C_{p,q}$ which depends only on p and q such that $I_{p,q}(f) \le C_{p,q}\left(|f(0)| + K_{p,q}(f)\right)$, for all $f \in H(\mathbb{D})$.

We shall also prove the following result for $1 < p < 2$.

Theorem 1.3. *If $1 < p < 2$, $p < q \le \infty$ and $f \in \mathscr{B}(p,q)$, then $J_{p,q}(f) < \infty$, where*

$$\begin{aligned} J_{p,q}(f) &\overset{def}{=} \left(\int_0^1 M_p^q(r,f)\left(\log\frac{2}{1-r}\right)^{-q/p}\frac{dr}{1-r}\right)^{1/q}, \quad \textit{if } q < \infty, \\ J_{p,\infty}(f) &\overset{def}{=} \sup_{0<r<1} M_p(r,f)\left(\log\frac{2}{1-r}\right)^{-1/p}. \end{aligned} \tag{1.9}$$

Furthermore, there exists a positive constant $C_{p,q}$ which depends only on p and q such that $J_{p,q}(f) \le C_{p,q}\left(|f(0)| + K_{p,q}(f)\right)$.

Taking $q = p$ in Theorem 1.2 and $q = 2$ in Theorem 1.3, we have the following result.

Theorem 1.4. *(i) If* $2 < p < \infty$ *and* $f \in \mathcal{D}^p_{p-1}$ *then*

$$\int_{\mathbb{D}} |f(z)|^p \left(\log \frac{2}{1-|z|}\right)^{-p/2} \frac{dA(z)}{1-|z|} < \infty. \tag{1.10}$$

(ii) If $1 < p < 2$ *and* $f \in \mathscr{B}(p,2)$*, then*

$$\int_0^1 M_p^2(r,f) \left(\log \frac{2}{1-r}\right)^{-2/p} \frac{dr}{1-r} < \infty. \tag{1.11}$$

It is well known that, for every p, the Hardy space H^p is contained in the Bergman space A^{2p}. This is also true for the spaces $\mathcal{D}^p_{p-1}$, that is, we have

$$\mathcal{D}^p_{p-1} \subset A^{2p}, \quad 0 < p < \infty.$$

The situation is different for the spaces $\mathscr{B}(p,2)$, we shall prove in Theorem 3.3 that

$$\mathscr{B}(p,2) \subset A^{2p} \iff p \geq 1.$$

Except for $p = 2$, all the inclusions in (1.2), (1.3) and (1.4) are strict. However, Baernstein, Girela and Peláez (cf. [4]) have recently proved, that for every $p \in (0,\infty)$, an analytic univalent function in the unit disk belongs to the Hardy space H^p if and only if it belongs to the Dirichlet space $\mathcal{D}^p_{p-1}$. Our main result in Section 8 asserts that this is not true for the spaces $\mathscr{B}(p,2)$. If $0 < p < 1/2$ then any analytic univalent function in the unit disk belongs belongs to H^p and, hence, also to $\mathscr{B}(p,2)$. However, we shall prove in Theorem 8.1 that:

- (i) If $\frac{1}{2} \leq p < 2$ then there exists an analytic univalent function in $\mathbb{D}$ which belongs to $\mathscr{B}(p,2) \setminus H^p$.
- (ii) If $2 < p < \infty$ then there exists an analytic univalent function in $\mathbb{D}$ which belongs to $H^p \setminus \mathscr{B}(p,2)$.

We shall close the paper with Section 9 where we shall use some of the results we have stated so far to study the Carleson measures for the spaces $\mathscr{B}(p,2)$ and $\mathcal{D}^p_{p-1}$.

Let us close this section saying that, as usual, throughout the paper C_p, $C_{p,q}$, ... will denote positive constants depending only on the displayed parameters but not necessarily the same at distinct occurrences. Also, $U \asymp V$ will mean that there is a constant $C > 0$ such that $(1/C)V \leq U \leq CV$.

2. Two lemmas and simple proofs of Theorem 1.1 and Theorem C

Using the closed graph theorem, (1.3) yields the following result.

Lemma 2.1. *If* $2 < p < \infty$ *then there is a constant* C_p *depending only on* p *such that*

$$\|f\|_{H^p} \leq C_p \left(|f(0)| + \left(\int_0^1 (1-r) M_p^2(r,f')\, dr \right)^{1/2} \right), \quad \text{for all } f \in H(\mathbb{D}). \tag{2.1}$$

Theorem 1.1 can be proved using this Lemma and arguing as in the proof of Theorem 2 (a) of [17].

Proof of Theorem 1.1. Take $p \in (2, \infty)$ and $f \in \mathscr{B}(p, \infty)$. Assume, without loss of generality, that $f(0) = 0$. For $0 < r < 1$, set $f_r(z) = f(rz)$, $(z \in \mathbb{D})$. Applying Lemma 2.1 to f_r, $(0 < r < 1)$, and using that $f \in \mathscr{B}(p, \infty)$, we obtain

$$\begin{aligned} M_p^2(r,f) &\le C_p \int_0^1 (1-\rho) M_p^2(r\rho, f')\, d\rho \le C \int_0^1 \frac{(1-\rho)}{(1-r\rho)^2}\, d\rho \\ &= C \left(\int_0^r \frac{(1-\rho)}{(1-r\rho)^2}\, d\rho + \int_r^1 \frac{(1-\rho)}{(1-r\rho)^2}\, d\rho \right), \quad 0 < r < 1. \end{aligned} \tag{2.2}$$

Since $r\rho < \rho$ and $r\rho < r$, $(0 < r, \rho < 1)$, (2.2) implies

$$\begin{aligned} M_p^2(r,f) &\le C \left(\int_0^r \frac{1}{1-\rho}\, d\rho + \frac{1}{(1-r)^2} \int_r^1 (1-\rho)\, d\rho \right) \\ &= \mathrm{O}\left(\log \frac{1}{1-r} \right), \quad \text{as } r \to 1. \end{aligned} \tag{2.3}$$

This finishes the proof. □

Lemma 2.1 does not extend to the case $p = \infty$, as is shown by the example $f(z) = \left(\log \frac{2}{1-z} \right)^\alpha$ with $0 < \alpha < 1/2$. However, we have:

Lemma A. ([28, Lemma 3]). If $f \in H(\mathbb{D})$, and $K_{\infty,2}(f) < \infty$, then $f \in BMOA$, and there is an absolute constant C such that

$$\|f\|_{BMOA} \le C \left(|f(0)| + K_{\infty,2}(f) \right). \tag{2.4}$$

Theorem C can be deduced using this lemma and arguing as in the proof of Theorem 1.1.

3. Some embedding theorems and other basic results on the spaces $\mathscr{B}(p,q)$

It is well known (see Theorem 8.20 in p. 215 of Vol. I of [39]) that if $f \in H(\mathbb{D})$ is given by a power series with Hadamard gaps and $0 < p < \infty$, then $M_2(r,f) \asymp M_p(r,f)$. On the other hand, an easy calculation implies that if f is an analytic function in $\mathbb{D}$ which is given by a power series with Hadamard gaps,

$$f(z) = \sum_{k=1}^\infty a_k z^{n_k} \quad (z \in \mathbb{D}) \quad \text{with } n_{k+1} \ge \lambda n_k, \quad \text{for all } k \quad (\lambda > 1),$$

then there are at most $C_\lambda = \log_\lambda 2 + 1$ of the $n_k' s$ in the set $I(n) = \{j \in \mathbb{N} : 2^n \le j < 2^{n+1}\}$, $n = 0, 1 \dots$. Thus, using Theorem 1 of [26] we obtain:

Lemma 3.1. *Let $0 < p, q < \infty$. If f is an analytic function in $\mathbb{D}$ which is given by a power series with Hadamard gaps,*

$$f(z) = \sum_{k=1}^\infty a_k z^{n_k} \quad (z \in \mathbb{D}) \quad \text{with } n_{k+1} \ge \lambda n_k, \quad \text{for all } k \quad (\lambda > 1),$$

then

$$f \in \mathscr{B}(p,q) \Leftrightarrow f \in \mathscr{B}(2,q) \Leftrightarrow \sum_{k=0}^\infty |a_k|^q < \infty.$$

In particular, if f is as in the statement of Lemma 3.1 (taking $q = p$) we have that

$$f \in \mathcal{D}^p_{p-1} \Leftrightarrow \sum_{k=0} |a_k|^p < \infty, \tag{3.1}$$

this and the well known result (see, e. g., chapter V in vol. I of [39]) that for Hadamard gap series as above we have, for $0 < p < \infty$,

$$f \in H^p \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} |a_k|^2 < \infty, \tag{3.2}$$

inmediately imply that $\mathcal{D}^p_{p-1} \neq H^p$ if $p \neq 2$. On the other hand, Lemma 3.1 and (3.2) imply that if f is as in the statement of Lemma 3.1 then

$$f \in \mathscr{B}(p,2) \Leftrightarrow f \in H^p \Leftrightarrow \sum_{k=0} |a_k|^2 < \infty. \tag{3.3}$$

However, in section 8 we shall prove the following result.

Theorem 3.2. *If $0 < p < \infty$ and $p \neq 2$, then $\mathscr{B}(p,2) \neq H^p$.*

By a theorem of Hardy and Littlewood [19] (see also Theorem 5.6 of [9] and [25] or [36] for a simple proof), for every p, the Hardy space H^p is contained in the Bergman space A^{2p} and the exponent $2p$ cannot be improved. Using (1.4) we deduce that, if $0 < p \leq 2$, then $\mathcal{D}^p_{p-1} \subset A^{2p}$. Actually, this is also true for $p > 2$. Thus, we have:

$$\mathcal{D}^p_{p-1} \subset A^{2p}, \quad 0 < p < \infty. \tag{3.4}$$

This is a particular case of Theorem 2.1 of [6] and follows from the work of Flett [12, 13]. In view of this result and (1.2), it is natural to ask whether the inclusion $\mathscr{B}(p,2) \subset A^{2p}$ $(0 < p < \infty)$ holds. Our next result asserts that the answer is affirmative if and only $p \geq 1$.

Theorem 3.3. *(a) If $1 \leq p < \infty$, then $\mathscr{B}(p,2) \subset A^{2p}$.*
(b) If $0 < p < 1$, then $\mathscr{B}(p,2) \not\subset A^{2p}$.

The following result is essentially due to Hardy and Littlewood and can be proved by modifying the proof of Theorem 5.9 in [9] and will be used in the proof of Theorem 3.3.

Lemma 3.4. *For $0 < p < q \leq \infty$, there exists a positive constant C_{pq} depending only on p and q such that for each $f \in H(\mathbb{D})$ and each $r \in (0,1)$ we have*

$$M_q(r,f) \leq C_{pq} M_p\left(\frac{1+r}{2}, f\right)(1-r)^{\frac{1}{q}-\frac{1}{p}}. \tag{3.5}$$

Proof of Theorem 3.3. Suppose first that $1 \leq p < \infty$, and $f \in \mathscr{B}(p,2)$. Using lemma 4.2.7 of [38], we see that if suffices to prove that

$$\int_0^1 I_{2p}(r,f')(1-r)^{2p}\,dr < \infty. \tag{3.6}$$

Using Lemma 3.4, we see that there exists $C_p > 0$ such that

$$I_{2p}(r,f') \leq C_p \frac{M_p^{2p}(\varrho,f')}{1-r}, \quad \varrho = \frac{1+r}{2},\ 0 < r < 1. \tag{3.7}$$

Furthermore, the condition $f \in \mathscr{B}(p,2)$ implies that

$$M_p(r,f') = \mathrm{o}\left(\frac{1}{1-r}\right), \quad \text{as} \quad r \to 1^-. \tag{3.8}$$

Putting together (3.7) and (3.8), we deduce that there exists $r_0 \in (0,1)$ such that

$$\begin{aligned}\int_{r_0}^1 I_{2p}(r,f')(1-r)^{2p}\,dr &\le C_p \int_{r_0}^1 M_p^{2p}(\varrho,f')(1-r)^{2p-1}\,dr \\ &\le C_p \int_{\frac{1+r_0}{2}}^1 M_p^{2p}(r,f')(1-r)^{2p-1}\,dr \\ &\le C \int_{\frac{1+r_0}{2}}^1 M_p^{2}(r,f')(1-r)\,dr < \infty.\end{aligned}$$

This finishes the proof of part (a).

Now we turn to part (b). Let $0 < p < 1$ and take

$$f(z) = \frac{1}{(1-z)^{1/p}\left(\log \frac{2e^{2/p}}{1-z}\right)^{1/2p}}, \quad z \in \mathbb{D}. \tag{3.9}$$

Since $z \to \log \frac{2e^{2/p}}{1-z}$ is a conformal map from the unit disk onto a domain D which is contained in $\{z \in \mathbb{C} : \mathrm{Re}(z) > 2/p, |\,\mathrm{Im}(z)| < \frac{\pi}{2}\}$, we have that

$$\left|\log \frac{2e^{2/p}}{1-z}\right| \le \log\left|\frac{2e^{2/p}}{1-z}\right| + \frac{\pi}{2} \le \pi \log\left|\frac{2e^{2/p}}{1-z}\right| \le \pi \log \frac{2e^{2/p}}{1-|z|}, \quad z \in \mathbb{D}.$$

Then it follows that

$$\begin{aligned} I_{2p}(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{dt}{|1-re^{it}|^2\left|\log \frac{2e^{2/p}}{1-re^{it}}\right|} \\ &\ge \frac{1}{2\pi^2}\left(\int_0^{2\pi}\frac{dt}{|1-re^{it}|^2}\right)\left(\log \frac{2e^{2/p}}{1-r}\right)^{-1} \\ &= \frac{1}{\pi(1-r^2)\log\frac{2e^{2/p}}{1-r}}, \quad 0<r<1.\end{aligned}$$

Thus,

$$\int_{\mathbb{D}} |f(z)|^{2p}\,dA(z) = \int_0^1 I_{2p}(r,f)\,dr = \int_0^1 \frac{dr}{\pi(1-r^2)\log\frac{2e^{2/p}}{1-r}} = \infty,$$

consequently $f \notin A^{2p}$.

Now we shall prove that $f \in \mathscr{B}(p,2)$. Bearing in mind (3.9) and the fact that the function $x \mapsto x^p \log \frac{2e^{p/2}}{x}$ is increasing in $(0,2)$, we deduce that

$$\begin{aligned} I_p(r,f') &\le C_p \int_0^{2\pi} \frac{dt}{|1-re^{it}|^{1+p}\left|\log \frac{2e^{2/p}}{1-re^{it}}\right|^{1/2}} \\ &\le C_p \left(\int_0^{2\pi} \frac{dt}{|1-re^{it}|^{1+p/2}}\right)\left((1-r)^p \log \frac{2e^{2/p}}{1-r}\right)^{-1/2} \\ &\le C_p \frac{1}{(1-r)^p \left(\log \frac{2e^{2/p}}{1-r}\right)^{1/2}}. \end{aligned}$$

Since $\frac{1}{p} > 1$, we have

$$\int_0^1 (1-r) M_p^2(r,f')\, dr \le C_p \int_0^1 \frac{dr}{(1-r)\left(\log \frac{2e^{2/p}}{1-r}\right)^{1/p}} < \infty.$$

This finishes the proof. □

4. Decomposition theorems I

Let $\varphi\colon (0,1] \mapsto [0,\infty)$ be a continuous function. Then φ is called a *normal weight* (cf. [33, 34]) if the following conditions are satisfied:

(L) there exists a constant $\alpha > 0$ such that the function $\varphi(x)/x^\alpha$ $(0 < x < 1)$ is almost increasing;

(U) there exists a constant $\beta > 0$ such that the function $\varphi(x)/x^\beta$ $(0 < x < 1)$ is almost decreasing.

(A nonnegative real function $\varphi(x)$ is almost increasing (cf. [5]) if there is a constant $C > 0$ such that $x < y$ implies $\varphi(x) \le C\varphi(y)$. An almost decreasing function is defined similarly). It is easily checked that the function

$$\varphi(x) = x^\alpha \left(\log \frac{2}{x}\right)^s \tag{4.1}$$

is a normal weight provided that $\alpha > 0$ and $-\infty < s < \infty$. On the other hand, the weight

$$\varphi(x) = \left(\log \frac{2}{x}\right)^{-s},$$

where $s > 0$, satisfies condition (U) but not (L).

The space $H(p,q,\varphi)$ $(0 < p, q \le \infty)$, introduced in [34] $(q = \infty)$ and [27] $(q < \infty)$, consists of those $f \in H(\mathbb{D})$ for which the function $F(r) = \varphi(1-r)M_p(r,f)$ belongs to the space $L^q(dr/(1-r))$. The norm in $H(p,q,\varphi)$ is given by

$$\|f\|_{H(p,q,\varphi)} = \|F\|_{L^q(dr/(1-r))}.$$

Notice that $f \in \mathscr{B}(p,q) \Leftrightarrow f' \in H(p,q,\varphi)$, with $\varphi(x) = x$. Spaces $H(p,q,\varphi)$ with non-normal weights were considered in [33, 27, 29, 30, 31].

For a function $f(z) = \sum_{n=0}^\infty a_n z^n$ analytic in $\mathbb{D}$, define the polynomials $\Delta_j f$:

$$\Delta_j f(z) = \sum_{k=2^j}^{2^{j+1}-1} a_k z^k, \quad \text{for } j \ge 1,$$

$$\Delta_0 f(z) = a_0 + a_1 z.$$

Theorem D. ([27, Theorem 2.1]). Let φ be a normal weight, $f \in H(\mathbb{D})$ and $1 < p < \infty$, $0 < q \le \infty$. Then $f \in H(p,q,\varphi)$ if and only if the sequence $\{\varphi(2^{-j})\|\Delta_j f\|_{H^p}\}_{j=0}^{\infty}$ belongs to ℓ^q. Moreover we have

$$\|f\|_{H(p,q,\varphi)} \asymp \left\|\{\varphi(2^{-j})\|\Delta_j f\|_{H^p}\}\right\|_{\ell^q}.$$

The following assertion is easily deduced from Theorem D and the fact that the sequence of functions z^n, $n \ge 0$, is a Shauder basis in H^p, $1 < p < \infty$.

Corollary A. ([27, Theorem 3.1]). With the above hypotheses, $f' \in H(p,q,\varphi)$ if and only if the sequence $\{2^j \varphi(2^{-j})\left\|\Delta_j f\right\|_p\}_{j=0}^{\infty}$ belongs to ℓ^q. Moreover we have

$$|f(0)| + \|f'\|_{H(p,q,\varphi)} \asymp \left\|\{2^j\varphi(2^{-j})\|\Delta_j f\|_{H^p}\}\right\|_{\ell^q}. \tag{4.2}$$

In particular,

$$|f(0)| + K_{p,q}(f) \asymp \left\|\{\|\Delta_j f\|_{H^p}\}\right\|_{\ell^q}, \quad 1 < p < \infty,\ 0 < q \le \infty. \tag{4.3}$$

As a special case of Corollary A we have the following result.

Corollary 4.1. Let $1 < p < \infty$. A function $f \in H(\mathbb{D})$ belongs to $\mathscr{B}(p,\infty)$ if and only if $\sup_j \|\Delta_j f\|_{H^p} < \infty$. Moreover, we have

$$|f(0)| + \sup_{0<r<1} (1-r) M_p(r,f')\, dr \asymp \sup_{j\ge 0} \|\Delta_j f\|_{H^p}. \tag{4.4}$$

5. Decomposition theorems II

Let

$$\lambda_0 = 0, \quad \text{and} \quad \lambda_n = 2^{2^n} \text{ for } n \ge 1. \tag{5.1}$$

For a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ let

$$\widetilde{\Delta}_n f(z) = \sum_{k=\lambda_n}^{\lambda_{n+1}-1} a_k z^k, \quad \text{for } n \ge 0.$$

We have

$$\widetilde{\Delta}_n f(z) = \sum_{j=2^n}^{2^{n+1}-1} \Delta_j f(z), \quad \text{for } n \ge 1,$$

and

$$\widetilde{\Delta}_0 f(z) = \Delta_0 f(z) + \Delta_1 f(z).$$

The spaces $\mathscr{G}_{\mathbf{p}}(\beta)$. For $0 < p \le \infty$ and $\beta > 0$, we define $\mathscr{G}_p(\beta)$ to be the space of those $f \in H(\mathbb{D})$ which satisfy (1.5), i.e., those for which

$$\|f\|_{\mathscr{G}_p(\beta)} \stackrel{\text{def}}{=} \sup_{r<1} M_p(r,f) \left(\log \frac{2}{1-r}\right)^{-\beta} < \infty.$$

Equivalently: $\mathscr{G}_p(\beta) = H(p,\infty,\varphi)$, where

$$\varphi(x) = \left(\log \frac{2}{x}\right)^{-\beta}, \quad 0 < x < 1.$$

We have the following characterization of the spaces $\mathscr{G}_p(\beta)$.

Theorem 5.1. *Let $1 < p < \infty$ and $f \in H(\mathbb{D})$. Then $f \in \mathscr{G}_p(\beta)$ if and only if*

$$\sup_n 2^{-n\beta} \|\widetilde{\Delta}_n f\|_{H^p} < \infty.$$

Moreover

$$\|f\|_{\mathscr{G}_p(\beta)} \asymp \sup_n 2^{-n\beta} \big\|\widetilde{\Delta}_n f\big\|_{H^p}.$$

For the proof we need two lemmas.

Lemma 5.2. ([27, Lemma 3.1]). *If $g(z) = \sum_{j=m}^n a_j z^j$, $m < n$, then*

$$r^n \|g\|_{H^p} \le M_p(r, g) \le r^m \|g\|_{H^p}, \quad 0 < r < 1.$$

Lemma 5.3. *If $\beta > 0$, then*

$$\sum_{n=0}^{\infty} 2^{n\beta} r^{\lambda_n} \le C \left(\log \frac{2}{1-r} \right)^{\beta}, \quad 0 < r < 1,$$

where C depends only on β.

Proof. We can assume that $r \ge 3/4$. Then choose $m \ge 2$ so that $r_{m-1} \le r \le r_m$, where $r_m = 1 - 1/\lambda_m$. Then

$$\sum_{n=0}^{\infty} 2^{n\beta} r^{\lambda_n} \le \sum_{n=0}^{m} 2^{n\beta} + \sum_{n=m+1}^{\infty} 2^{n\beta} e^{-\lambda_n/\lambda_m}.$$

Since $\lambda_{n+1}/\lambda_n \ge 4$, for $n \ge 1$, whence $\lambda_n/\lambda_m \ge 4^{n-m}$ for $n \ge m$, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} 2^{n\beta} r^{\lambda_n} &\le C_\beta 2^{m\beta} + \sum_{j=1}^{\infty} 2^{(j+m)\beta} e^{-4^j} \\ &\le (C_\beta + C'_\beta) 2^{m\beta}, \end{aligned}$$

where

$$C'_\beta = \sum_{j=1}^{\infty} 2^{j\beta} e^{-4^j} < \infty.$$

This gives the desired result because

$$\log \frac{2}{1-r} \ge \log \frac{2}{1-r_{m-1}}$$

and

$$\log \frac{2}{1-r_{m-1}} \asymp 2^m \quad (m \ge 1).$$

Proof of Theorem 5.1. Let $n \ge 0$. By the Riesz projection theorem and Lemma 5.2, we have

$$M_p(r, f) \ge c M_p(r, \widetilde{\Delta}_n f) \ge c r^{\lambda_{n+1}} \|\widetilde{\Delta}_n f\|_{H^p}, \quad 0 < r < 1,$$

where $c > 0$ is a constant depending only on p. Hence, by taking $r = 1 - 1/\lambda_{n+1}$,

$$\sup_{r<1} M_p(r, f) \left(\log \frac{2}{1-r} \right)^{-\beta} \ge c' \sup_n 2^{-n\beta} \|\widetilde{\Delta}_n f\|_{H^p}.$$

In the other direction, assuming that $M := \sup_n 2^{-n\beta}\|\widetilde{\Delta}_n f\|_{H^p} < \infty$, we have

$$\begin{aligned} M_p(r,f) &\le \sum_{n=0}^{\infty} M_p(r,\widetilde{\Delta}_n f) \\ &\le \sum_{n=0}^{\infty} r^{\lambda_n}\|\widetilde{\Delta}_n f\|_{H^p} \\ &\le M\sum_{n=0}^{\infty} 2^{n\beta} r^{\lambda_n}. \end{aligned}$$

Now Lemma 5.3 concludes the proof. □

6. Decomposition theorems III

In this section we consider the space $\mathscr{G}_p^q(\beta)$ consisting of those $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathscr{G}_p^q(\beta)} := \left\{ \int_0^1 M_p^q(r,f) \left(\log\frac{2}{1-r}\right)^{-\beta q} \frac{dr}{1-r} \right\}^{1/q} < \infty. \tag{6.1}$$

Therefore $\mathscr{G}_p^q(\beta) = H(p,q,\varphi)$, where

$$\varphi(x) = \left(\log\frac{2}{x}\right)^{-\beta}, \quad 0 < x < 1.$$

The space $\mathscr{G}_p^q(\beta)$ is non-trivial if and only if $\beta > 1/q$ and can be characterized as follows.

Theorem 6.1. *Let $1 < p < \infty$, $0 < q < \infty$ and $\beta > 1/q$. A function $f \in H(\mathbb{D})$ belongs to $\mathscr{G}_p^q(\beta)$ if and only if*

$$\sum_{n=0}^{\infty} 2^{-n(\beta q-1)}\|\widetilde{\Delta}_n f\|_{H^p}^q < \infty. \tag{6.2}$$

Furthermore,

$$\|f\|_{\mathscr{G}_p^q(\beta)} \asymp \left(\sum_{n=0}^{\infty} 2^{-n(\beta q-1)}\|\widetilde{\Delta}_n f\|_{H^p}^q\right)^{1/q}.$$

The following lemma will be needed in the proof of Theorem 6.1.

Lemma 6.2. *Let*

$$h(r) = \sum_{n=0}^{\infty} b_n r^{\lambda_n},$$

where $b_n \ge 0$, and $h(r) < \infty$ for $r \in [0,1)$. Let $0 < q < \infty$, and $\alpha > 1$. Then

$$\int_0^1 h(r)^q \left(\log\frac{2}{1-r}\right)^{-\alpha} \frac{dr}{1-r} \asymp \sum_{n=0}^{\infty} 2^{-n(\alpha-1)} b_n^q. \tag{6.3}$$

The constants involved in this estimate depends only on α and q.

Proof. Let L and R denote the left hand side and the right hand side of (6.3), respectively. Let $r_n = 1 - 1/\lambda_n$ for $n \geq 1$, and $r_0 = 0$. Then:

$$\begin{aligned} L &= \sum_{n=0}^{\infty} \int_{r_{n+1}}^{r_{n+2}} h(r)^q \left(\log \frac{2}{1-r} \right)^{-\alpha} \frac{dr}{1-r} \\ &\geq \sum_{n=0}^{\infty} \int_{r_{n+1}}^{r_{n+2}} (b_n r^{\lambda_{n+1}})^q \left(\log \frac{2}{1-r_{n+2}} \right)^{-\alpha} \frac{dr}{1-r} \\ &\geq c \sum_{n=0}^{\infty} b_n^q 2^{-n\alpha} \int_{r_{n+1}}^{r_{n+2}} \frac{dr}{1-r} \\ &\geq c' \sum_{n=0}^{\infty} b_n^q 2^{-n\alpha} 2^n \end{aligned}$$

(c and c' are positive constants). This proves the inequality $L \geq c'R$.

In proving the reverse estimate we first consider the case $q \leq 1$. Let

$$\psi(r) = \left(\log \frac{2}{1-r} \right)^{-\alpha} \frac{1}{1-r}.$$

Using the inequality $(a+b)^q \leq a^q + b^q$ $(a,\, b \geq 0)$ we get

$$L \leq \sum_{n=0}^{\infty} b_n^q \int_0^1 r^{q\lambda_n} \psi(r)\, dr.$$

Hence we have to prove that

$$\int_0^1 r^{q\lambda_n} \psi(r)\, dr \leq C 2^{-n(\alpha-1)}. \tag{6.4}$$

Choose $\eta \in (0,1)$ so that the function ψ increases on $[\eta, 1)$. If $n \geq 1$ is such that $r_n = 1 - 1/\lambda_n \leq \eta$, then

$$\int_0^1 r^{q\lambda_n} \psi(r)\, dr \leq \int_0^1 \psi(r)\, dr \leq C 2^{-n(\alpha-1)},$$

where we have used the fact that ψ is integrable on $(0,1)$. Let $r_n > \eta$. Then we write

$$\begin{aligned} \int_0^1 r^{q\lambda_n} \psi(r)\, dr &= \left(\int_0^{\eta} + \int_{\eta}^{r_n} + \int_{r_n}^1 \right) r^{q\lambda_n} \psi(r)\, dr \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We have

$$I_1 \leq \frac{M}{q\lambda_n + 1} \leq CM 2^{-n(\alpha-1)},$$

where $M = \max_{[0,\eta]} \psi < \infty$. In the case of I_2 we use the fact that ψ is increasing in $[\eta, 1)$ to obtain

$$I_2 \leq \psi(r_n) \int_{\eta}^1 r^{q\lambda_n}\, dr \leq \frac{\lambda_n \left(\log(2\lambda_n) \right)^{-\alpha}}{q\lambda_n + 1} \leq C 2^{-n\alpha}.$$

Finally,

$$I_3 \leq \int_{r_n}^1 \psi(r)\, dr = \frac{(\log(2\lambda_n))^{1-\alpha}}{\alpha - 1} = \frac{((2^n+1)\log 2)^{1-\alpha}}{\alpha - 1}.$$

This complete the proof in the case $q \leq 1$.

In order to discuss the case $q>1$ we introduce the measure $d\mu$ on $(0,1)$ by

$$d\mu(r)=\left(\log\frac{2}{1-r}\right)^{-1}\frac{dr}{1-r}.$$

Then we change the notation by putting $\alpha=q\gamma+1$ ($\gamma>0$), and $2^{-n(\alpha-1)/q}b_n=2^{-n\gamma}b_n=c_n$ to rewrite the inequality $L\le CR$ as

$$\left\{\int_0^1|H(r)|^q\left(\log\frac{2}{1-r}\right)^{-q\gamma}d\mu(r)\right\}^{1/q}\le C_q\left\{\sum_{n=0}^{\infty}|c_n|^q\right\}^{1/q}, \tag{6.5}$$

where

$$H(r)=\sum_{n=0}^{\infty}2^{n\gamma}c_nr^{\lambda_n}.$$

Here we assume that c_n are complex numbers, and interpret the case $q=\infty$ in (6.5) as

$$\sup_{0<r<1}|H(r)|\left(\log\frac{2}{1-r}\right)^{-\gamma}\le C_\infty\sup_{n\ge0}|c_n|.$$

This inequality holds because of Lemma 5.3. On the other hand, by the first part of the proof, (6.5) holds for $q=1$. To deduce the validity of (6.5) for $1<q<\infty$, it is enough to consider the operator

$$\{c_n\}_0^\infty\longmapsto H(\cdot)\left(\log\frac{2}{1-\cdot}\right)^{-\gamma}$$

and apply the Riesz–Thorin interpolation theorem. We are done. □

Proof of Theorem 6.1. By the Riesz projection theorem and Lemma 5.2 we have, for $1<p<\infty$,

$$M_p(r,f)\ge cr^{\lambda_{n+1}}\|\widetilde{\Delta}_nf\|_{H^p},$$

where $c>0$ is independent of f and n. On the other hand, by Lemma 5.2 and Minkowski's inequality, it follows that

$$M_p(r,f)\le\sum_{n=0}^{\infty}r^{\lambda_n}\|\widetilde{\Delta}_nf\|_{H^p}.$$

Now Lemma 6.2 concludes the proof. □

7. Proofs of Theorems 1.2 and 1.3 and some related results

Proof of Theorem 1.2. Take p and q with $2<p<\infty$ and $2<q\le\infty$. It follows from (2.1) and (4.3) that

$$\|f\|_{H^p}^2\le C_p\left(|f(0)|+K_{p,2}(f)\right)^2\le C_p\sum_{j=0}^{\infty}\|\Delta_jf\|_{H^p}^2, \tag{7.1}$$

for all $f\in H(\mathbb{D})$.

Take $f\in\mathscr{B}(p,q)$. Applying (7.1) to $\widetilde{\Delta}_nf$, we obtain

$$\|\widetilde{\Delta}_nf\|_{H^p}^2\le C_p\sum_{j\in I_n}\|\Delta_jf\|_{H^p}^2, \tag{7.2}$$

where,

$$I_n = \begin{cases} \{j\colon 2^n \le j \le 2^{n+1}-1\}, & n \ge 1,\\ \{0,1\}, & n = 0.\end{cases}$$

Suppose first that $q=\infty$. Since $f \in \mathscr{B}(p,\infty)$, (7.2) and Corollary 4.1 imply

$$2^{-n}\|\widetilde{\Delta}_n f\|_{H^p}^2 \le C_p \sup_j \|\Delta_j f\|_{H^p}^2 < \infty$$

and then Theorem 5.1 and (4.3) yield that $I_{p,\infty}(f) \le C_{p,\infty}\left(|f(0)| + K_{p,\infty}(f)\right) < \infty$.

Suppose now that $2 < q < \infty$. Using (7.2) and applying Jensen's inequality for the convex function $x \mapsto x^{q/2},\ x > 0$, we obtain

$$\begin{aligned} 2^{-n(q/2-1)}\|\widetilde{\Delta}_n f\|_{H^p}^q &\le C_p 2^n \left(2^{-n}\sum_{j\in I_n}\|\Delta_j f\|_{H^p}^2\right)^{q/2} \\ &\le C_p 2^n \sum_{j\in I_n} 2^{-n}\|\Delta_j f\|_{H^p}^q = C_p \sum_{j\in I_n}\|\Delta_j f\|_{H^p}^q, \quad \text{for all } n. \end{aligned}$$

Then it follows that

$$\sum_{n=0}^{\infty} 2^{-n(q/2-1)}\|\widetilde{\Delta}_n f\|_{H^p}^q \le C_p \sum_{j=0}^{\infty}\|\Delta_j f\|_{H^p}^q,$$

and Theorem 6.1 and (4.3) yield $I_{p,q}(f) \le C_{p,q}\left(|f(0)| + K_{p,q}(f)\right) < \infty$ finishing the proof. □

Proof of Theorem 1.3. Take p and q with $1 < p < 2$ and $p < q \le \infty$. Using (1.4) and the closed graph theorem and (4.3), we obtain

$$||f||_{H^p}^p \le C_p\left(|f(0| + K_{p,p}(f)\right)^p \le C_p \sum_{j=0}^{\infty}||\Delta_j f||_{H^p}^p, \tag{7.3}$$

for all $f \in H(\mathbb{D})$.

Take $f \in \mathscr{B}(p,q)$. Applying (7.3) to $\widetilde{\Delta}_n f$, it follows that

$$\|\widetilde{\Delta}_n f\|_{H^p}^p \le C_p \sum_{j\in I_n}\|\Delta_j f\|_{H^p}^p. \tag{7.4}$$

Suppose first that $q=\infty$. Using (7.4) and (4.3) we obtain

$$2^{-n}\|\widetilde{\Delta}_n f\|_{H^p}^p \le C_p \sum_{j\in I_n} 2^{-n}\|\Delta_j f\|_{H^p}^p \le \sup_j \|\Delta_j f\|_{H^p}^p \asymp \left(|f(0)| + K_{p,\infty}(f)\right)^p.$$

Then Theorem 5.1 gives $J_{p,\infty}(f) \le C_p\left(|f(0)| + K_{p,\infty}(f)\right) < \infty$.

Consider now the case $q < \infty$. Since $p < q$, using Hölder's inequality with the exponents $q/(q-p)$ and q/p, (7.4) implies

$$||\tilde{\Delta}_n f||_{H^p}^q \le C_{p,q}\left(\sum_{j\in I(n)}||\Delta_j f||_{H^p}^p\right)^{\frac{q}{p}} \le C_{p,q}2^{n(\frac{q}{p}-1)}\sum_{j\in I(n)}||\Delta_j f||_{H^p}^q.$$

and, hence,

$$\sum_{n=0}^{\infty} 2^{-n(\frac{q}{p}-1)}||\tilde{\Delta}_n f||_{H^p}^q \le C_{p,q}\sum_{j=0}^{\infty}||\Delta_j f||_{H^p}^q.$$

Then Theorem 6.1 and (4.3) yield $J_{p,q}(f) \leq C_{p,q}\left(|f(0)| + K_{p,q}(f)\right) < \infty$. $\square$

Our next result shows that theorems 1.2 and 1.3 are sharp.

Theorem 7.1. *(a) Let $0 < p, \varepsilon < \infty$ and $2 < q < \infty$. Then there exists $f \in \mathscr{B}(p,q)$ such that*

$$\int_0^1 M_p^q(r,f)\left(\log\frac{2}{1-r}\right)^{-\frac{q}{2}+\varepsilon}\frac{dr}{1-r} = \infty. \tag{7.5}$$

(b) Let $0 < p, \varepsilon < \infty$ and $p < q < \infty$. Then there exists $f \in \mathscr{B}(p,q)$ such that

$$\int_0^1 M_p^q(r,f)\left(\log\frac{2}{1-r}\right)^{-\frac{q}{p}+\varepsilon}\frac{dr}{1-r} = \infty. \tag{7.6}$$

Proof of Theorem 7.1.

(a) Let p, ε and q be as in part (a). Take γ with $0 < \gamma < \min\left\{\frac{1}{2} - \frac{1}{q}, \frac{\varepsilon}{q}\right\}$ and set

$$f(z) = \sum_{j=1}^{\infty} \frac{1}{j^{1/q+\gamma}} z^{2^j}, \quad z \in \mathbb{D}. \tag{7.7}$$

Since f is given by a power series with Hadamard gaps, using Lemma 3.1 and the fact that $\sum_{j=1}^{\infty} j^{-1-q\gamma} < \infty$, we deduce that $f \in \mathscr{B}(p,q)$.

On the other hand, a direct calculation gives that

$$M_2(r,f) \asymp \left(\log\frac{2}{1-r}\right)^{\frac{1}{2}-\frac{1}{q}-\gamma} \quad \text{as} \quad r \to 1^-.$$

Now since $M_p(r,f) \asymp M_2(r,f)$, we deduce that

$$\begin{aligned}
&\int_0^1 M_p^q(r,f)\left(\log\frac{2}{1-r}\right)^{-\frac{q}{2}+\varepsilon}\frac{dr}{1-r} \\
&\asymp \int_0^1 \left(\log\frac{2}{1-r}\right)^{-1-q\gamma+\varepsilon}\frac{dr}{1-r} = \infty,
\end{aligned}$$

since $0 < \gamma < \frac{\varepsilon}{q}$.

(b) Let p, ε and q as in part (b). Without loss of generality, assume that $0 < \varepsilon < \frac{q}{p} - 1$. Take $\gamma > 0$ such that $\frac{1}{q} < \gamma < \frac{1+\varepsilon}{q}$ and set $a = \max\{2e, 2e^{2\gamma}\}$. Now define

$$f(z) = \frac{1}{(1-z)^{1/p}\left(\log\frac{a}{1-z}\right)^{\gamma}}, \quad z \in \mathbb{D}. \tag{7.8}$$

Since the function $x \mapsto x\left(\log\frac{a}{x}\right)^{2\gamma}$ is an increasing function in $(0,2)$ we deduce that

$$\begin{aligned}
I_p(r,f') &\leq C_p \int_0^{2\pi} \frac{dt}{|1-re^{it}|^{1+p}\left|\log\frac{a}{1-re^{it}}\right|^{p\gamma}} \\
&\leq C_p \left(\int_0^{2\pi}\frac{dt}{|1-re^{it}|^{1+p/2}}\right)\left((1-r)\left(\log\frac{a}{1-r}\right)^{2\gamma}\right)^{-p/2} \qquad (7.9)\\
&\leq C_p \frac{1}{(1-r)^p\left(\log\frac{a}{1-r}\right)^{p\gamma}},
\end{aligned}$$

consequently, since $q\gamma > 1$,

$$\int_0^1 M_p^q(r,f')(1-r)^{q-1}\,dr \le C_p \int_0^1 \frac{1}{(1-r)\left(\log \frac{a}{1-r}\right)^{q\gamma}}\,dr < \infty.$$

Notice that $p\gamma < 1$ and recall that $\gamma < \frac{1+\varepsilon}{q}$, then, arguing as in part (b) of Theorem 3.3, we deduce that

$$I_p(r,f) \ge C\left(\log \frac{a}{1-r}\right)^{1-p\gamma},$$

and then

$$\begin{aligned}&\int_0^1 M_p^q(r,f)\left(\log\frac{2}{1-r}\right)^{-\frac{q}{p}+\varepsilon}\frac{dr}{1-r}\\ &\ge C\int_0^1\left(\log\frac{2}{1-r}\right)^{-q\gamma+\varepsilon}\frac{dr}{1-r} = \infty.\end{aligned}$$

□

8. Univalent functions

A complex-valued function defined in $\mathbb{D}$ is said to be *univalent* if it is analytic and one-to-one there. We refer to [10] and [32] for the theory of these functions. Throughout the paper, $\mathcal{U}$ will stand for the class of all univalent functions in $\mathbb{D}$. Sometimes it is useful to consider certain normalized subclasses of $\mathcal{U}$ such as the class S and the class S_0:

$$S = \{f \in \mathcal{U} : f(0) = 0, f'(0) = 1\},$$

$$S_0 = \{f \in \mathcal{U} : f \text{ is zero-free in } \mathbb{D}, f(0) = 1\}.$$

Theorem 1 of [4] asserts that

$$\mathcal{U} \cap \mathcal{D}^p_{p-1} = \mathcal{U} \cap H^p, \quad 0 < p < \infty.$$

In view of (1.2), (1.3) and (1.4), it is natural to ask whether or not the univalent functions in $\mathscr{B}(p,2)$ coincide with those in H^p, $0 < p < \infty$. Obviously, the answer to this question is affirmative if $p = 2$. If $0 < p < \frac{1}{2}$ the answer is also affirmative by (1.2), since it is well known (see e.g Theorem 3.16 of [9]) that $\mathcal{U} \subset H^p$, for all $p < 1/2$. However we shall prove the following theorem.

Theorem 8.1. *(a) If $\frac{1}{2} \le p < 2$ then there exists $f \in \mathcal{U}$ such that $f \in \mathscr{B}(p,2) \setminus H^p$.*
(b) If $2 < p < \infty$ then there exists $f \in \mathcal{U}$ such that $f \in H^p \setminus \mathscr{B}(p,2)$.

Before getting into the proof of this result we shall use it to prove Theorem 3.2.

Proof of Theorem 3.2. Theorem 8.1 implies the conclusion of Theorem 3.2 for $p \ge 1/2$, $p \ne 2$. On the other hand, Theorem 3.2 for $p < 1/2$ follows from Theorem 3.3 and the relation $H^p \subset A^{2p}$, $0 < p < \infty$. □

In order to prove Theorem 8.1 we shall need several preliminary results.

Lemma 8.2. *For $0 < \alpha \le 1$, define*

$$Q_\alpha(z) = \frac{1}{(1-z)^\alpha \log \frac{2e^{1/\alpha}}{1-z}}, \quad z \in \mathbb{D}. \tag{8.1}$$

Then:

$$Q_\alpha \in \mathcal{U}. \tag{8.2}$$

$$M_\infty(r, Q_\alpha) = \frac{1}{(1-r)^\alpha \log \frac{2e^{1/\alpha}}{1-r}}, \quad 0 < r < 1. \tag{8.3}$$

The proof is similar to that of Lemma 4 in [4] and will be omitted.

Lemma 8.3. *For $0 < p < \infty$ there exists a positive constant C_p such that*

$$\left(\int_0^1 (1-r)^{\frac{2}{p}-1} M_\infty^2(r,f)\, dr\right)^{1/2} \le C_p \left(|f(0)| + K_{p,2}(f)\right), \tag{8.4}$$

for all $f \in H(\mathbb{D})$.

Proof of Lemma 8.3. Take $f \in H(\mathbb{D})$. Theorem 5.6 of [9] with $p = \infty$, $a = 2$ and $b = \frac{2}{p} - 1$ yields

$$\int_0^1 (1-r)^{\frac{2}{p}-1} M_\infty^2(r,f)\, dr \le C_p \left(|f(0)|^2 + \int_0^1 (1-r)^{\frac{2}{p}+1} M_\infty^2(r,f')\, dr\right). \tag{8.5}$$

Now, using Lemma 3.4 with $q = \infty$, we obtain

$$(1-r)^{\frac{2}{p}+1} M_\infty^2(r,f') \le C_p (1-r) M_p^2\left(\frac{1+r}{2}, f'\right), \quad 0 < r < 1.$$

Using this in (8.5) and making the change of variable $\rho = (1+r)/2$, yields

$$\int_0^1 (1-r)^{\frac{2}{p}-1} M_\infty^2(r,f)\, dr \le C_p \left(|f(0)|^2 + \int_0^1 (1-\rho) M_p^2(\rho,f')\, d\rho\right),$$

which is equivalent to (8.4). $\Box$

Proof of Theorem 8.1.
(a) Let $\frac{1}{2} \le p < 2$. Set

$$f(z) = \frac{1}{\left((1-z)\log \frac{2e}{1-z}\right)^{1/p}}, \quad z \in \mathbb{D}.$$

In Theorem 3 of [4], it is proved that $f \in \mathcal{U}$ and that $f \notin H^p$. Now, arguing as in (7.9) we deduce that there exists $C_p > 0$ and $r_0 \in (0,1)$ such that

$$M_p(r,f') \le \frac{C_p}{(1-r)\left(\log \frac{2e}{1-r}\right)^{1/p}}, \quad r_0 < r < 1.$$

Then it follows that $f \in \mathscr{B}(p,2)$. This finishes the proof of part (a).

(b) Let $2 < p < \infty$ and take $\alpha = 2/p$ in Lemma 8.2. Then we have that $Q_{2/p} \in \mathcal{U}$, where,

$$Q_{2/p}(z) = \frac{1}{(1-z)^{2/p} \log \frac{2e^{p/2}}{1-z}}, \quad z \in \mathbb{D}.$$

Obviously we also have that $f = Q_{2/p}^{1/2} \in \mathcal{U}$. Moreover, by (8.3),

$$\int_0^1 M_\infty^p(r,f)\,dr < \infty,$$

consequently we deduce that $f \in H^p$ (see p. 127 of [32]).

On the other hand, (8.3) also implies that

$$\int_0^1 (1-r)^{2/p-1} M_\infty^2(r,f)\,dr = \infty,$$

which, together with Lemma 8.3, yields that $f \notin \mathscr{B}(p,2)$. This finishes the proof. □

Even though we have proved that, for $p \in [1/2,2) \cup (2,\infty)$, the H^p-univalent functions are not the same as the $\mathscr{B}(p,2)$-univalent functions, next we are going to present another caracterization of H^p-univalent functions valid for all $p \in (0,\infty)$. For this purpose we need to introduce some new spaces of analytic functions in $\mathbb{D}$. With other notation, these spaces were considered in [26].

If $0 < p < \infty$ and $f(z) = \sum_{n=0}^\infty a_n z^n \in H(\mathbb{D})$, we define

$$\begin{aligned}
&E_p(r,f) \stackrel{\text{def}}{=} \int_{|z|<r} |f'(z)|^2 |f(z)|^{p-2} dA(z), \quad 0<r<1,\\
&A(r,f) \stackrel{\text{def}}{=} \int_{|z|<r} |f'(z)|^2\,dA(z) = \pi \sum_{n=1}^\infty n|a_n|^2 r^{2n}, \quad 0<r<1,\\
&P(r,f) \stackrel{\text{def}}{=} \sum_{n=0}^\infty |a_n| r^n, \quad 0<r<1.
\end{aligned} \tag{8.6}$$

We note that $A(r,f)$ is the area of the image of the disk $\{z : |z| < r\}$ under f, counting multiplicities. The quantity $E_p(r,f)$ plays a very important role in questions concerning the integral means of f, because the following identity of Hardy-Stein (see Chapter 5 of [32]):

$$r\frac{d}{dr} I_p(r,f) = \frac{p^2}{2\pi} E_p(r,f), \quad 0<r<1. \tag{8.7}$$

We shall use the notation $f \in H_\alpha^p$, $p>0$, $\alpha>0$, whenever

$$||f||_{H_\alpha^p}^p \stackrel{\text{def}}{=} \int_0^1 E_p(r,f)(1-r)^{\alpha-1}\,dr < \infty. \tag{8.8}$$

By (8.7), it is clear that $H^p = H_1^p$. We shall write $f \in S_\alpha^p$, $p>0$, $\alpha>0$, whenever

$$||f||_{S_\alpha^p}^p \stackrel{\text{def}}{=} \int_0^1 A^{p/2}(r,f)(1-r)^{\alpha-1}\,dr < \infty. \tag{8.9}$$

It is obvious that H_α^2 coincides with S_α^2, for all $\alpha>0$. If $p \neq 2$ (see Proposition 1 and Theorem 2 of [26]) the following inclusions are proved.

Theorem E. Let $\alpha > 0$. Then

$$\begin{aligned}
&H_\alpha^p \subset S_\alpha^p, \quad 0<p<2,\\
&S_\alpha^p \subset H_\alpha^p, \quad 2<p<\infty.
\end{aligned} \tag{8.10}$$

Furthermore, both inclusions in Theorem E are strict (see p. 312 of [26]). However, we shall prove the following result.

Theorem 8.4. *Let $p > 0$ and $\alpha > 0$. Then*

$$H^p_\alpha \cap \mathcal{U} = S^p_\alpha \cap \mathcal{U}. \tag{8.11}$$

For $\alpha = 1$, Theorem 8.4 and Theorem 1 of [4] yield the following result.

Corollary 8.5. Let $p > 0$. Then

$$\mathcal{D}^p_{p-1} \cap \mathcal{U} = H^p \cap \mathcal{U} = S^p_1 \cap \mathcal{U}. \tag{8.12}$$

The following lemma is an extension of Theorem 5.1 of [32] and will be needed in the proof of Theorem 8.4.

Lemma 8.6. *Suppose that $0 < p < \infty$, $\alpha > 0$ and $f \in S$. If*

$$\int_0^1 (1-r)^{\alpha-1} M^p_\infty(r,f)\, dr < \infty,$$

then $f \in H^p_\alpha$.

Proof. Suppose that $f \in S$, then making the change of variable $w = f(z)$ we have that

$$\begin{aligned} &\frac{p^2}{2\pi} E_p(r,f) = \frac{p^2}{2\pi} \int_{|z|<r} |f(z)|^{p-2} |f'(z)|^2 \, dA(z) \\ &\le \frac{p^2}{2\pi} \int_{|w| \le M_\infty(r,f)} |w|^{p-2} \, dA(w) = p^2 \int_0^{M_\infty(r,f)} t^{p-1} \, dt = p\, M^p_\infty(r,f) \end{aligned} \tag{8.13}$$

and then the lemma follows. □
Proof of Theorem 8.4.
Let $\alpha > 0$. If $0 < p \le 2$ by (8.10) it suffices to show that

$$S^p_\alpha \cap S \subset H^p_\alpha \cap S.$$

Let $f \in S^p_\alpha \cap S$. By Proposition 2 of [26] and the inequality

$$M_\infty(r,f) \le P(r,f), \quad 0 < r < 1,$$

we have that

$$\int_0^1 (1-r)^{\alpha-1} M^p_\infty(r,f)\, dr < \infty.$$

Then Lemma 8.6 implies that $f \in H^p_\alpha$.
If $2 \le p < \infty$, by (8.10), it suffices to show that

$$H^p_\alpha \cap S \subset S^p_\alpha \cap S.$$

Let $f \in H^p_\alpha \cap S$. The identity $E_2(r,f) = A(r,f)$, (8.13) and Lemma 8.6 imply

$$\begin{aligned} \int_0^1 (1-r)^{\alpha-1} A^{p/2}(r,f)\, dr &= \int_0^1 (1-r)^{\alpha-1} E_2^{p/2}(r,f)\, dr \\ &\le \pi^{p/2} \int_0^1 (1-r)^{\alpha-1} M^p_\infty(r,f)\, dr < \infty, \end{aligned}$$

that is, $f \in S^p_\alpha$. This finishes the proof. □

It is well known (see, e.g., [15]) that

$$BMOA \subset \mathscr{B}, \quad \text{and} \quad H^\infty \subset BMOA \subset \cap_{0<p<\infty} H^p.$$

However, $H^\infty \not\subset \cup_{0<p<2}\mathcal{D}^p_{p-1}$. Indeed, Vinogradov proves in Theorem 3.11 of [35] that there exists a Blaschke product B such that $B \notin \cup_{0<p<2}\mathcal{D}^p_{p-1}$. Then, in view of (1.2), (1.3) and (1.4), it is natural to ask whether or not $BMOA$ or H^∞ are contained in $\mathscr{B}(p,2)$, $2<p<\infty$. We de not know the answer, but we can prove the following result.

Theorem 8.7.

$$BMOA \cap \mathcal{U} \subset (\cap_{0<p<\infty}\mathscr{B}(p,2)) \cap \mathcal{U}. \tag{8.14}$$

Proof. Let $f \in BMOA \cap \mathcal{U}$. Bearing in mind (1.2), it is clear that it suffices to prove that $f \in \mathscr{B}(p,2)$ for $1 \le p < \infty$. Since $BMOA \subset \mathscr{B}$, we have

$$M_\infty(r,f) = \mathrm{O}\left(\log\frac{1}{1-r}\right), \quad \text{as } r \to 1^-.$$

Using Theorem 1 of [3] (with $\alpha = 1/2$), we deduce that

$$M_1(r,f') = \mathrm{O}\left(\left(\frac{1}{1-r}\right)^{1/2}\right), \quad r \to 1^-,$$

and, then, bearing in mind that $BMOA \subset \mathscr{B}$, we deduce that

$$M_p^p(r,f') = \mathrm{O}\left(\left(\frac{1}{1-r}\right)^{p-\frac{1}{2}}\right), \quad r \to 1^-, \quad p \ge 1,$$

which implies

$$\int_0^1 (1-r)M_p^2(r,f')\,dr \le C\int_0^1 (1-r)^{\frac{1}{p}-1}\,dr < \infty, \quad p \ge 1.$$

Hence, $f \in \mathscr{B}(p,2)$, for all $p \ge 1$. This finishes the proof. □

9. Carleson measures for the spaces $\mathscr{B}(p,2)$

If E is a measurable subset of the unit circle $\mathbb{T} = \partial\mathbb{D}$, we write $|E|$ for the Lebesgue measure of E. If $I \subset \mathbb{T}$ is an interval, the *Carleson square* $S(I)$ is defined as

$$S(I) = \left\{re^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \le r < 1\right\}.$$

Carleson [7] (see also Theorem 9.3 of [9]) proved that if $0 < p < \infty$ and μ is a positive Borel measure in $\mathbb{D}$ then $H^p \subset L^p(d\mu)$ if and only if there exists a positive constant C such that

$$\mu\big(S(I)\big) \le C|I|, \text{ for every interval } I \subset \mathbb{T}. \tag{9.1}$$

The measures μ which satisfy this condition will be called *classical Carleson measures.*

A positive Borel measure μ in $\mathbb{D}$ is said to be a Carleson measure for $\mathcal{D}^p_{p-1}$ (respectively, for $\mathscr{B}(p,2)$) if $\mathcal{D}^p_{p-1} \subset L^p(d\mu)$ (respectively, if $\mathscr{B}(p,2) \subset L^p(d\mu)$).

The Carleson measures for the spaces $\mathcal{D}^p_{p-1}$, $0 < p \le 2$, were characterized by Vinogradov [35] and Wu [37] who proved that they are precisely the classical Carleson measures. Wu conjectured in [37] that this remains true for $2 < p < \infty$, but this conjecture has been recently disproved by Girela and Peláez (cf. [18]).

Our main object in this section is studying the Carleson measures for the spaces $\mathscr{B}(p,2)$. We can prove the following result.

Theorem 9.1. *Let $2 < p < \infty$. A positive Borel measure μ in $\mathbb{D}$ is a Carleson measure for $\mathscr{B}(p,2)$ if and only if μ is a classical Carleson measure.*

Proof. Let μ be a classical Carleson measure, then $H^p \subset L^p(d\mu)$. Now, since $p > 2$, $\mathscr{B}(p,2) \subset H^p$. Consequently, we have that $\mathscr{B}(p,2) \subset L^p(d\mu)$ and, hence, μ is a Carleson measure for $\mathscr{B}(p,2)$.

Assume now that μ is a Carleson measure for $\mathscr{B}(p,2)$. By the closed graph Theorem, there exists a positive constant C_p such that

$$\left(\int_{\mathbb{D}} |f(z)|^p \, d\mu(z)\right)^{1/p} \leq C_p \left(|f(0)| + K_{p,2}(f)\right). \tag{9.2}$$

In order to prove that μ is a classical Carleson measure, it suffices to prove that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)}{|1-\overline{a}z|^2} \, d\mu(z) < \infty, \tag{9.3}$$

(see Lemma 3.3 in p. 239 of [14]). For $a \in \mathbb{D}$, take the "test" function

$$f_a(z) = \left(\frac{1-|a|^2}{(1-\overline{a}z)^2}\right)^{1/p} \qquad z \in \mathbb{D}.$$

Differentiating and bearing in mind (9.2) we deduce that

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{(1-|a|^2)}{|1-\overline{a}z|^2} \, d\mu(z) \\
&\leq C_p \left(|f_a(0)| + K_{p,2}(f)\right)^p \\
&\leq C_p \left(|f_a(0)|^p + K^p_{p,2}(f)\right) \\
&\leq C_p(1-|a|^2) \left[1 + \left(\frac{2|a|}{p}\right)^p \left(\int_0^1 (1-r) \left(\int_0^{2\pi} \frac{dt}{|1-\overline{a}re^{it}|^{2+p}}\right)^{2/p} dr\right)^{p/2}\right] \\
&\leq C_p(1-|a|^2) \left[1 + C'_p \left(\int_0^1 \frac{(1-r)}{(1-|a|r)^{2+2/p}} \, dr\right)^{p/2}\right] \\
&\leq C_p(1-|a|^2) \left[1 + C'_p \left(\int_0^1 \frac{1}{(1-|a|r)^{1+2/p}} \, dr\right)^{p/2}\right] \\
&\leq C_p < \infty.
\end{aligned}$$

So we have (9.3). This finishes the proof. □

We observe that in the last argument we do not use the condition $p > 2$, so we can state the following result.

Proposition 9.2. If $0 < p \leq 2$ and μ is a positive Borel measure in $\mathbb{D}$ which is a Carleson measure for $\mathscr{B}(p,2)$ then it is a classical Carleson measure.

It is natural to ask whether the converse is true. We do not know the answer but, by analogy with what happens for the spaces $\mathcal{D}^p_{p-1}$, $2 < p < \infty$, we conjecture that the answer is negative.

Using Theorem 1.4(ii), we can obtain a condition on μ which is sufficient for being a Carleson measure for $\mathscr{B}(p,2)$, $1 < p < 2$.

Theorem 9.3. *Suppose that $1 < p < 2$ and that μ is a positive Borel measure in $\mathbb{D}$ for which there exists a positive constant C such that*

$$\mu(S(I)) \leq \frac{C|I|}{\log^{2/p}\frac{2}{|I|}}, \quad \text{for all intervals } I \subset \mathbb{T}, \tag{9.4}$$

then μ is a Carleson measure for $\mathscr{B}(p,2)$.

Also, using Theorem 1.4(i), we can obtain a condition on μ which is sufficient for being a Carleson measure for $\mathcal{D}^p_{p-1}$, $2 < p < \infty$. This result improves Proposition 2.2 of [18].

Theorem 9.4. *Suppose that $2 < p < \infty$ and that μ is a positive Borel measure in $\mathbb{D}$ for which there exists a positive constant C such that*

$$\mu(S(I)) \leq \frac{C|I|}{\log^{p/2}\frac{2}{|I|}}, \quad \text{for all intervals } I \subset \mathbb{T}, \tag{9.5}$$

then μ is a Carleson measure for $\mathcal{D}^p_{p-1}$.

In the proofs of these results we shall use arguments similar to those used to characterize the Carleson measures for the Bergman spaces which can be found in Section 2.10 of [11].

Throughout this section, ϱ will denote the pseudo-hyperbolic metric in the unit disk:

$$\varrho(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|, \quad z,w \in \mathbb{D}.$$

The *pseudohyperbolic disk* of (pseudohyperbolic) center a and radius r ($a \in \mathbb{D}$, $0 < r < 1$) is the set $\Delta(a,r) = \{z \in \mathbb{D} : \varrho(a,z) < r\}$. It coincides with the Euclidean disk whose (Euclidean) radius and center are (see p. 40 of [11]):

$$R = \frac{1-|a|^2}{1-r^2|a|^2} r, \qquad c = \frac{1-r^2}{1-r^2|a|^2} a. \tag{9.6}$$

Arguing as in pp. 65-66 of [11] we can easily deduce the following result.

Lemma 9.5. *Suppose that $0 < p < \infty$ and $0 < r < 1$. Let μ be a positive Borel measure in $\mathbb{D}$.*

(i) If there exists a positive constant C such that (9.4) holds, then there exists a positive constant C_r which depends only on r such that

$$\mu(\Delta(a,r)) \leq \frac{C_r(1-|a|)}{\log^{2/p}\frac{2}{1-|a|}}, \quad a \in \mathbb{D}. \tag{9.7}$$

(ii) If there exists a positive constant C such that (9.5) holds, then there exists a positive constant C_r which depends only on r such that

$$\mu(\Delta(a,r)) \leq \frac{C_r(1-|a|)}{\log^{p/2}\frac{2}{1-|a|}}, \quad a \in \mathbb{D}. \tag{9.8}$$

Proof of Theorem 9.3. First of all, let us notice that using Theorem 1.4(ii) we see that:

$$\text{If } f \in \mathscr{B}(p,2),\ 1<p<2, \text{ then } \int_{\mathbb{D}} |f(z)|^p \left(\log\frac{2}{1-|z|}\right)^{-2/p} \frac{dA(z)}{1-|z|} < \infty \tag{9.9}$$

Take $p \in (1,2)$ and let μ be a positive Borel measure in $\mathbb{D}$ which satisfies (9.4). Take $r \in (0,1)$. Using Lemma 9.5, we see that there exists $C_r > 0$ such that (9.7) holds.

Using Lemma 12 in p. 62 of [11], we see that there exist a sequence $\{a_k\}_{k=1}^{\infty}$ of points of $\mathbb{D}$ and an integer N such that

$$\mathbb{D} = \cup_{k=1}^{\infty} \Delta(a_k, r) \tag{9.10}$$

and no point $z \in \mathbb{D}$ belongs to more than N of the disks $\Delta(a_k, R)$, where $R = (1+r)/2$.

Using Lemma 13 in p. 63 of [11], we deduce that there exists $C_R > 0$ such that, for any $f \in H(\mathbb{D})$,

$$|f(z)|^p \le \frac{C_R}{m\left(\Delta(a,R)\right)} \int_{\Delta(a,R)} |f(\zeta)|^p dA(\zeta), \quad z \in \Delta(a,r), \quad a \in \mathbb{D}. \tag{9.11}$$

We remark that here and throughout the paper, if F is a measurable subset of $\mathbb{C}$, $m(F)$ will denote its area (two-dimensional Lebesgue measure).

Take $f \in \mathscr{B}(p,2)$. Using (9.10) and (9.11), if follows that

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^p d\mu(z) \le& \sum_{k=1}^{\infty} \int_{\Delta(a_k,r)} |f(z)|^p d\mu(z) \\ \le& C_R \sum_{k=1}^{\infty} \frac{\mu(\Delta(a_k,r))}{m(\Delta(a_k,R))} \int_{\Delta(a_k,R)} |f(\zeta)|^p dA(\zeta). \end{aligned} \tag{9.12}$$

Bearing in mind Lemma 9.5(i) and the fact that $m(\Delta(a,R)) \asymp (1-|a|^2)$, $a \in \mathbb{D}$ (with constants depending only on R), we see that

$$\frac{\mu(\Delta(a_k,r))}{m(\Delta(a_k,R))} \le \frac{C}{(1-|a_k|)\log^{2/p} \frac{2}{1-|a_k|}}.$$

Using this in (9.12), bearing in mind Lemma 3 in p. 41 of [11] and the fact that no point $z \in \mathbb{D}$ lies in more than N of the disks $\Delta(a_k, R)$, and, using (9.9) we obtain

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^p d\mu(z) \le& C \sum_{k=1}^{\infty} \int_{\Delta(a_k,R)} |f(z)|^p \left(\log \frac{2}{1-|z|}\right)^{-2/p} \frac{dA(z)}{1-|z|} \\ \le& C \int_{\mathbb{D}} |f(z)|^p \left(\log \frac{2}{1-|z|}\right)^{-2/p} \frac{dA(z)}{1-|z|} \\ <& \infty. \end{aligned}$$

Thus, $f \in L^p(d\mu)$. This finishes the proof. □

Theorem 9.4 can be proved with the same arguments used in the proof of Theorem 9.3 (using Theorem 1.4(i) instead of (9.9)). We omit the details.

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